

# On Pfaffian and determinant of one type of skew centrosymmetric matrices

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## Abstract

This paper is dedicated to compute Pfaffian and determinant of one type of skew centrosymmetric matrices in terms of general number sequence of second order.

**Key words:** Pfaffian; determinant; skew centrosymmetric matrix

## 1 Introduction

The *determinant* is one of the basic parameters in matrix theory. For an  $n$ -square matrix  $A$ , it is defined by

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}.$$

The *Pfaffian* which is intimately related to determinant, was introduced by Cayley, denoted by  $Pf(A)$ , is defined by

$$Pf(A) = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^n a_{\pi(2i-1)\pi(2i)}.$$

The Pfaffian of a skew symmetric matrix is a quantity closely related to the determinant. Cayley's Theorem says that the square of the Pfaffian of a matrix is equal to the determinant of the matrix. In other words, for an  $n$ -square skew-symmetric matrix  $A$ ,

$$\det(A) = [Pf(A)]^2.$$

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A matrix  $A$  is a *centrosymmetric matrix* if  $A = JAJ^{-1}$  where  $J$  is anti-diagonal matrix whose anti-diagonal entries are one and others are zero. If  $A = -AJA^{-1}$ , it is said to be *skew-centrosymmetric matrix*. This matrix family have wide applications in many fields of science such as, numerical solution of certain differential equations, digital signal processing, information theory, statistics, linear systems theory, some Markov processes and so on (see [1, 2, 3, 4, 5, 6]).

This paper highlights the close connections among the Pfaffian, the determinant and general number sequences of second order.

## 2 Pfaffian and Skew centrosymmetric matrices

**Definition 1** Let us define  $n$ -square matrices  $A_n = [a_{i,j}]$  and  $B_n = [b_{i,j}]$  in the given form

$$[a_{i,j}] = \begin{cases} a, & \text{for } j = i + 1 \\ -a, & \text{for } i = j + 1 \\ 0, & \text{otherwise} \end{cases}, \quad [b_{i,j}] = \begin{cases} (-1)^{i+1}b, & \text{for } i + j = n + 1 \\ 0, & \text{otherwise} \end{cases}$$

where  $1 \leq i, j \leq n$ .

**Definition 2** Let us define  $2 \times 2$ -block matrices as below.

$$\mathcal{F}_n = \begin{pmatrix} A_k & B_k \\ (-1)^k B_k & A_k \end{pmatrix} \text{ and } \mathcal{G}_n = \begin{pmatrix} A_k & -B_k \\ (-1)^{k+1} B_k & A_k \end{pmatrix}.$$

For example, for  $n = 10$ , the  $n$ -square ( $n = 2k$ ,  $k$  is any positive integer) skew centrosymmetric matrix  $\mathcal{F}_n$  will be in the following form:

$$\mathcal{F}_{10} = \left( \begin{array}{ccccc|ccccc} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\ -a & 0 & a & 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & -a & 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 & a & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -b & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & -a & 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & -a & 0 & a & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & -a & 0 & a \\ -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \end{array} \right).$$

**Definition 3** Let us consider couple-recurrences given below.

$$\begin{aligned} f_n &= bg_{n-1} + a^2 f_{n-2} \quad \text{for } f_1 = b, \\ g_n &= -bf_{n-1} + a^2 g_{n-2} \quad \text{for } g_1 = -b. \end{aligned}$$

Then we have the following theorem:

**Theorem 4** For  $n = 2k$ ,

$$f_k = \text{Pf}(\mathcal{F}_n) \quad \text{and} \quad g_k = \text{Pf}(\mathcal{G}_n),$$

where  $f_{-1} = 0$ ,  $f_0 = 1$  and  $g_{-1} = 0$ ,  $g_0 = 1$ .

**Proof.** Let us prove the theorem by using mathematical induction method. For  $k = 1$ ,

$$\mathcal{F}_2 = \begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{G}_2 = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

In this case,

$$f_1 = \text{Pf}(\mathcal{F}_2) = b, \quad g_1 = \text{Pf}(\mathcal{G}_2) = -b.$$

Assume that the recurrences hold for all  $t \leq k$ . Then they hold for  $k = t + 1$ .

$$\mathcal{F}_{2t+2} = \left( \begin{array}{c|c} A_{t+1} & B_{t+1} \\ \hline (-1)^{t+1}B_{t+1} & A_{t+1} \end{array} \right) = \left( \begin{array}{c|cccc|c} 0 & a & 0 & \cdots & 0 & b \\ -a & & & & & 0 \\ 0 & & A_t & & -B_t & \vdots \\ \vdots & & (-1)^{t+1}B_t & & A_t & 0 \\ 0 & & & & & a \\ \hline -b & 0 & \cdots & 0 & -a & 0 \end{array} \right). \quad (1)$$

From the expansion formula along with  $2t + 2$  column of (1), it follows that

$$\text{Pf}(\mathcal{F}_{2t+2}) = b\text{Pf}(\mathcal{G}_{2t}) + a\text{Pf}(\mathcal{M}_{2t}) = bg_t + a\text{Pf}(\mathcal{M}_{2t}), \quad (2)$$

where

$$\mathcal{M}_{2t} = \left( \begin{array}{cc|cccc} 0 & a & 0 & \cdots & \cdots & 0 \\ -a & 0 & a & 0 & \cdots & 0 \\ \hline 0 & a & & & & \\ \vdots & 0 & & A_{t-1} & & B_{t-1} \\ \vdots & \vdots & & (-1)^{t-1}B_{t-1} & & A_{t-1} \\ 0 & 0 & & & & \end{array} \right) \quad (3)$$

From the expansion formula along with 1st row of (3), it follows that

$$\text{Pf}(\mathcal{M}_{2t}) = a\text{Pf}(\mathcal{F}_{2t-2}) = af_{t-1}. \quad (4)$$

From (2) and (4), we have

$$f_{t+1} = bg_t + a^2f_{t-1}.$$

The recurrence for  $g_{t+1}$  can be obtained in a similar manner. ■

**Corollary 5**  $f_n = (-1)^{n-1}bf_{n-1} + a^2f_{n-2}$  with  $f_{-1} = 0$  and  $f_1 = 1$ .

### 3 Determinant of the skew centrosymmetric matrix

In this section, we consider determinant of the matrix  $\mathcal{F}_n$  ( $n = 2k$ ). It is a well-known fact [3] for block matrices:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(AD - BC)$$

if it verifies  $AC = CA$ . Taking into account this property, determinant of the matrix  $\mathcal{F}_n$  is

$$|\mathcal{F}_n| = |\mathcal{T}_k| = \det \begin{pmatrix} -a^2 + b^2 & 0 & a^2 & & \\ 0 & -2a^2 + b^2 & 0 & \ddots & \\ a^2 & 0 & \ddots & \ddots & a^2 \\ & \ddots & \ddots & -2a^2 + b^2 & 0 \\ & & a^2 & 0 & -a^2 + b^2 \end{pmatrix}_{k \times k}.$$

Sogabe and El-Mikkawy [8] considered a fast block diagonalization of  $k$ -tridiagonal matrices using permutation matrices. Exploiting this method, we can rearrange the matrix  $\mathcal{T}_k$ .

**(i) For  $k$  is odd**

For this let us define the following matrices:

$$H_{\frac{k-1}{2}} = \begin{cases} -2a^2 + b^2, & \text{for } i = j \\ a^2, & \text{for } i = j + 1 \text{ and } j = i + 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$K_{\frac{k+1}{2}} = \begin{cases} -a^2 + b^2, & i = j = 1 \text{ and } i = j = \frac{k+1}{2} \\ -2a^2 + b^2, & i = j = 2(1) \frac{k-1}{2} \\ a^2, & \text{for } i = j + 1 \text{ and } j = i + 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$P^T \mathcal{F}_k P = \left( \begin{array}{c|c} H_{\frac{k-1}{2}} & 0 \\ \hline 0 & K_{\frac{k+1}{2}} \end{array} \right),$$

where permutation matrix  $P$  is determined by using method in [8]. Obviously,

$$\det(P^T \mathcal{T}_k P) = \det \mathcal{T}_k = \det \mathcal{F}_n = \det(H_{\frac{k-1}{2}}) \det(K_{\frac{k+1}{2}}).$$

**(ii) For  $k$  is even**

Let us define

$$N_{\frac{k}{2}} = \begin{cases} -a^2 + b^2, & i = j = \frac{k}{2} \\ -2a^2 + b^2, & i = j = 1(1)\frac{k}{2} - 1 \\ a^2, & \text{for } i = j + 1 \text{ and } j = i + 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q_{\frac{k}{2}} = \begin{cases} -a^2 + b^2, & i = j = 1 \\ -2a^2 + b^2, & i = j = 2(1)\frac{k}{2} \\ a^2, & \text{for } i = j + 1 \text{ and } j = i + 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$P^T \mathcal{F}_k P = \left( \begin{array}{c|c} N_{\frac{k}{2}} & 0 \\ \hline 0 & Q_{\frac{k}{2}} \end{array} \right).$$

Obviously,

$$\det(P^T \mathcal{T}_k P) = \det \mathcal{T}_k = \det \mathcal{F}_n = \det(N_{\frac{k}{2}}) \det(Q_{\frac{k}{2}}).$$

It can be seen that  $\det(N_{\frac{k}{2}}) = \det(Q_{\frac{k}{2}})$ .

El-Mikkawy [9] obtained determinant of tridiagonal matrix. That is,

$$v_i = \begin{vmatrix} d_1 & a_1 & 0 & \dots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & b_3 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{i-1} \\ 0 & \dots & 0 & b_i & d_i \end{vmatrix},$$

where  $v_i = d_i v_{i-1} - b_i a_{i-1} v_{i-2}$  for  $v_0 = 1$  and  $v_{-1} = 0$ . By exploiting [9] and Laplace expansion:

For  $k$  is even

$$\det(N_{\frac{k}{2}}) = \det(Q_{\frac{k}{2}}) = (-a^2 + b^2)w_{\frac{k}{2}-1} - a^4 w_{\frac{k}{2}-2}.$$

For  $k$  is odd

$$\begin{aligned} \det(K_{\frac{k+1}{2}}) &= (-a^2 + b^2)^2 w_{\frac{k-3}{2}} - 2a^4(-a^2 + b^2)w_{\frac{k-5}{2}} + a^8 w_{\frac{k-7}{2}}, \\ \det(H_{\frac{k-1}{2}}) &= w_{\frac{k-1}{2}}, \end{aligned}$$

where

$$w_i = \begin{vmatrix} -2a^2 + b^2 & a^2 & \dots & 0 \\ a^2 & -2a^2 + b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^2 \\ 0 & \dots & a^2 & -2a^2 + b^2 \end{vmatrix}.$$

Here  $w_i = (-2a^2 + b^2)w_{i-1} - a^4 w_{i-2}$  for  $w_0 = 1$  and  $w_{-1} = 0$ .

Consequently, for  $n = 2k$ ,

i) If  $k$  is odd,

$$\det \mathcal{F}_n = \det \mathcal{T}_k = w_{\frac{k-1}{2}} \left( (-a^2 + b^2)^2 w_{\frac{k-3}{2}} - 2a^4(-a^2 + b^2)w_{\frac{k-5}{2}} + a^8 w_{\frac{k-7}{2}} \right).$$

ii) If  $k$  is even,  $\det \mathcal{F}_n = \det \mathcal{T}_k = \left( (-a^2 + b^2)w_{\frac{k}{2}-1} - a^4 w_{\frac{k}{2}-2} \right)^2.$

## 4 Examples

Let us consider the matrix  $\mathcal{F}_n$  ( $n = 2k$ ). Then examples of the Pfaffians and the determinants are shown in Tables 1 and 2 respectively. Here  $F_n, P_n$  and  $J_n$  are  $n$ th Fibonacci, Pell and Jacobsthal numbers, respectively.

Table 1: Examples of the Pfaffians.

| $k$                        | $a = i, b = 1$         | $a = i, b = 2$         | $a = i\sqrt{2}, b = 1$ |
|----------------------------|------------------------|------------------------|------------------------|
|                            | $Pf(\mathcal{F}_{2k})$ | $Pf(\mathcal{F}_{2k})$ | $Pf(\mathcal{F}_{2k})$ |
| 1                          | $F_2 = 1$              | $P_2 = 2$              | $J_2 = 1$              |
| 2                          | $-F_3 = -2$            | $-P_3 = -5$            | $-J_3 = -3$            |
| 3                          | $-F_4 = -3$            | $-P_4 = -12$           | $-J_4 = -5$            |
| 4                          | $F_5 = 5$              | $P_5 = 29$             | $J_5 = 11$             |
| 5                          | $F_6 = 8$              | $P_6 = 70$             | $J_6 = 21$             |
| 6                          | $-F_7 = -13$           | $-P_7 = -169$          | $-J_7 = -43$           |
| 7                          | $-F_8 = -21$           | $-P_8 = -408$          | $-J_8 = -85$           |
| 8                          | $F_9 = 34$             | $P_9 = 985$            | $J_9 = 171$            |
| $\vdots$                   | $\vdots$               | $\vdots$               | $\vdots$               |
| $\equiv 0, 1(\text{mod}4)$ | $F_{k+1}$              | $P_{k+1}$              | $J_{k+1}$              |
| $\equiv 2, 3(\text{mod}4)$ | $-F_{k+1}$             | $-P_{k+1}$             | $-J_{k+1}$             |

Table 2: Examples of the determinants.

| $k$      | $a = i, b = 1$           | $a = i, b = 2$           | $a = i\sqrt{2}, b = 1$   |
|----------|--------------------------|--------------------------|--------------------------|
|          | $\det(\mathcal{F}_{2k})$ | $\det(\mathcal{F}_{2k})$ | $\det(\mathcal{F}_{2k})$ |
| 1        | $F_2^2$                  | $P_2^2$                  | $J_2^2$                  |
| 2        | $F_3^2$                  | $P_3^2$                  | $J_3^2$                  |
| 3        | $F_4^2$                  | $P_4^2$                  | $J_4^2$                  |
| 4        | $F_5^2$                  | $P_5^2$                  | $J_5^2$                  |
| 5        | $F_6^2$                  | $P_6^2$                  | $J_6^2$                  |
| 6        | $F_7^2$                  | $P_7^2$                  | $J_7^2$                  |
| 7        | $F_8^2$                  | $P_8^2$                  | $J_8^2$                  |
| 8        | $F_9^2$                  | $P_9^2$                  | $J_9^2$                  |
| $\vdots$ | $\vdots$                 | $\vdots$                 | $\vdots$                 |
| $t$      | $F_{t+1}^2$              | $P_{t+1}^2$              | $J_{t+1}^2$              |

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